New Investigation on the Spheroidal Wave Equations *

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Abstract

Changing the spheroidal wave equations into new Schrödinger's form, the super-potential expanded in the series form of the parameter $\alpha$are obtained in the paper. This general form of the super-potential makes it easy to get the ground eigenfunctions of the spheroidal equations. But the shape-invariance property is not retained and the corresponding recurrence relations of the form (4) could not be extended from the associated Legendre functions to the case of the spheroidal functions.

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Introduction

Spheroidal wave functions appear in many different context in physics and mathematics [1]-[4]. Their differential equations are

$$\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]+E+\alpha x^2-\frac{m^2}{1-x^2}\right]\Theta=0, x\in(-1,+1). \tag{1}$$

They just have one more term αx^2 than the spherical wave equations (the associated Legendre' equations). This extra term makes great difference in Equations: the spherical wave equations belong to the case of the confluent super-geometrical equations with one regular and one irregular singularities, whereas the spheroidal wave equations are the confluent Heun equations containing two regular and one irregular singularities. The extra singularity makes them one of the toughest problems for researchers to treat[1]-[3].

The spheroidal functions are the solutions of the Sturm-Liouville eigenvalue problem of Equation (1) with the natual condition Θ finite at the boundaries $x=\pm 1$. The eigenvalues are the allowable values $E_0,\ E_1,\ldots,E_n,\ldots$ of the parameter E; and the spheroidal wave functions are the corresponding eigenfunctions $\Theta_0,\Theta_1,\ldots,\Theta_n,\ldots$[1]-[3].

Though the spheroidal functions are the extension of the associated Legendre-functions $P_l^m(x)$, they stand in vivid contrast against the associated Legendre-functions. The associated Legendre-functions $P_l^m(x)$ have mane elegant properties:

1. When $m=0$, the Legendre-functions $P_n(x)=P_n^0(x)$ are polynomials of the independent variable x;

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2. Back to the variable θ with $x = \cos\theta$, the recursion relation among the Legendre-functions could be written as:

$$P_n(\cos\theta) = n^{-1}\left[-n\cos\theta - \sin\theta\frac{d}{d\theta}\right] P_{n-1}(\cos\theta),\ n = 1, 2, 3, \ldots, \tag{2}$$

so, all $P_n(\cos\theta)$ could be deduced from the ground function P_0 from the recursion relations

3. With the quantity m fixed, all the associated Legendre-functions $P_n^m(\cos\theta)$ could be derived from P_m^m by

$$\begin{aligned} P_n^m(\cos\theta) &= [(n+m)(n-m)]^{-\frac{1}{2}}\left[-n\cos\theta - \sin\theta\frac{d}{d\theta}\right] P_{n-1}^m(\cos\theta), \\ n &= m+1, m+2, m+3, \ldots, . \end{aligned} \tag{3}$$

4. With the quantity n fixed and the transformation $P_n^m = \frac{\Psi_n(\theta,m)}{\sin^{\frac{1}{2}}\theta}$, the recurrence relations between adjoint m of the associated Legendre-functions become

$$\begin{aligned} \Psi_n(\theta, m+1) &= [(n+m+1)(n-m)]^{-\frac{1}{2}} * \left[(m+\frac{1}{2})\cot\theta - \frac{d}{d\theta}\right] \Psi_n(\theta, m), \\ m &= 0, 1, 2, \ldots, n. \end{aligned} \tag{4}$$

These equations (4) are more useful than that of (3), they result in the relation

$$P_n^m(x) = \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m}. \tag{5}$$

The associated Legendre-functions $P_n^m(x)$ are related to the Legendre's functions by (5), which is the repeated forms of the relations (4). The significant meaning of these relations are that one only needs to know the Legendre's functions to obtain the associated Legendre's functions. So the two important works of the associated Legendre's equations lie in finding the ground state function of the Legendre's equations, that is, the case of $m = 0$ and the recurrence relations of the type of Eqs.(3) and (4). Eqs.(3) could be used to get the excited state eigenfunctions of the Legendre's equations; while Eqs.(4) make it possible to obtain the associated Legendre's functions from the Legendre's functions.

Like the spherical wave functions, the spheroidal wave functions are important functions applied extensively in many different branches in physics and mathematics [1]-[4], hence, it is natural for one to require whether or not the similar relations exist for the spheroidal functions. This is a long standing problem. However, it has actually been treated in the series papers[8]-[9]. In these papers, the supersymmetric quantum mechanics (SUSYQM) is first applied to study them, and new interesting results are obtained[6]-[10]: (1)the general form of the the ground eigenvalue and eigenfunction are given, which reduce to the ground eigenvalue $m(m+1)$ and the ground eigenfunction P_m^m when $\alpha = 0$; (2) the generalized recurrence relations like that of (3) are obtained for the spheroidal functions [10]. See references [6]-[10] for details. As stated before, the relations (4) are crucial for one to obtain the associated Legendre's functions from the Legendre's polynomials. Therefore, it is natural for one to

investigate whether the same kind recurrence relations could be extended to the case of the spheroidal functions. This is just what the present paper tries to do.

As done before, the supersymmetry quantum mechanics is used to deal with the problem. The recurrence relations turn out to be the shape-invariance relations for the corresponding differential equation. Thus, the spheroidal equations are first transformed into the Schrödinger equations. Then the super-potential W is introduced and solved in the series form of the parameter α. Thirdly, the ground state eigenfunction is obtained. Finally, the shape-invariance property is checked, and the recurrence relations are studied. In the following, the same steps will proceed too.

In use of the perturbation methods in super-symmetry quantum mechanics, it is necessary to rewrite the differential equations in the Schrödinger form. In the previous papers, the original form

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\alpha\cos^2\theta-\frac{m^2}{\sin^2\theta}\right]\Theta=-E\Theta \tag{6}$$

is used to study. It is obtained from Eq.(1) by the transformation $x=\cos\theta$ and the corresponding boundary conditions become Θ finite at $\theta=0,\ \pi$.

There are two ways to transform Eq.(6) or (1) into the forms of the Schrödinger equation. The first transformation is to change the eigenfunction through the transformation $\Theta=\frac{\Psi}{\sin^{\frac{1}{2}}\theta}$, and the Schrödinger's form becomes

$$\frac{d^2\Psi}{d\theta^2}+\left[\frac{1}{4}+\alpha\cos^2\theta-\frac{m^2-\frac{1}{4}}{\sin^2\theta}+E\right]\Psi=0. \tag{7}$$

The corresponding boundary conditions now are $\Psi(0)=\Psi(\pi)=0$.

In the series papers [6]-[10], the supersymmetric quantum mechanics (SUSYQM) is first applied to treat Eqs.(7). The focus is the super-potential W, which is determined by the potential

$$V(\theta,\alpha,m)=-\left[\frac{1}{4}+\alpha\cos^2\theta-\frac{m^2-\frac{1}{4}}{\sin^2\theta}\right]=W^2-\frac{dW}{d\theta}. \tag{8}$$

The ground function is connected with the super-potential W by

$$\Psi_0=N\exp\left[-\int Wd\theta\right]. \tag{9}$$

Hence, whenever the super-potential is known, the ground state function is known too [6]-[10]. By the perturbation methods in the super-symmetry quantum mechanics, it is the super-potential W that is expanded in the series form of the parameter α, that is, $W=\sum_{n=0}^{\infty}\alpha^nW_n$. By this method, new interesting results are obtained[6]-[10]: the first several terms of the the ground eigenvalue and eigenfunction are given, which reduce to the ground eigenvalue $m(m+1)$ and the ground eigenfunction P_m^m of the associated Lengdre' functions. Of course, these results are obtained through the corresponding terms for the super-potential W[6]-[8]:

$$W_0=-\left(m+\frac{1}{2}\right)\cot\theta,\ E_{00}=m(m+1). \tag{10}$$

$$W_1=\frac{\sin\theta\cos\theta}{2m+3} \tag{11}$$

$$W_2=\left[\frac{-\sin\theta\cos\theta}{(2m+3)^3(2m+5)}+\frac{\sin^3\theta\cos\theta}{(2m+3)^2(2m+5)}\right]. \tag{12}$$

Later the reference [9] generalized the results and obtained the general form for the super-potential $W = \sum_{n=0}^{\infty} W_n \alpha^n$ as

$$W_n = \sum_{k=1}^{n} \frac{\hat{a}_{n,k}}{(2m+3)^n} \sin\theta \cos^{2k+1}\theta, n = 1, 2, \ldots \tag{13}$$

where $\hat{a}_{n,k}$ could be easily determined step by step. The shape-invariance property is also proved in it. The form of the super-potential in (13) is a little different in form from that of the references [6]. The reference [10] directly generalized the form of the reference [6] and gived an alternative form of the super-potential as

$$W_n = \cos\theta \sum_{k=1}^{n} \tilde{a}_{n,k} \sin^{2k-1}\theta. \tag{14}$$

Our motivation is to investigate whether we could extend the relations of the type (4) to the cases of the spheroidal functions. In these recurrence relations, it is the integer m that is different. From the point of the super-symmetry quantum mechanics, the recurrence relations are oriented in the properties of the shape-invariance of the super-potential and are the relations between different eigen-functions of correspondingly different eigen-values. Therefore, the integer m should stand in the position of the eigen-values, the energies. Actually, this could not be met in Eq.(6) where it is the quantity E that stands as the eigen-value. However, there is another way to transform Eq.(6) to the Schrödinger form. Changing the independent variable θ to the new one $z = \lg\tan(\frac{\theta}{2})$, the new Schrödinger's form for the spheroidal wave equations is obtained as:

$$\frac{d^2\Theta}{dz^2} + \left[E\text{sech}^2 z + \alpha\text{sech}^2 z - \alpha\text{sech}^4 z - m^2\right]\Theta = 0. \tag{15}$$

First, one notices that the interval $(0, \pi)$ in the original variable θ now corresponds to the interval $(-\infty, +\infty)$ in the new variable z. Secondly, the boundary conditions now turn out as Θ finite at $z \to \pm\infty$. The most important thing is that Eq (15) makes the term containing the original eigenvalue E no longer in the position of the eigenvalue. Instead, it is the quantity $-m^2$ now that is the eigenvalue of Eq.(15). This is just what we want to obtain. It will beneficial for one to further investigate the kind relations 4 for the spheroidal functions.

In the above Eq.(15), the potential is

$$V(z, \alpha) = -\left[E\text{sech}^2 z + \alpha\text{sech}^2 z - \alpha\text{sech}^4 z\right]. \tag{16}$$

When $\alpha = 0$, Eq.(15) is just the form from the associated Legendre equations (6); its ground energy is $-m^2$, and the nodeless ground eigenfunction is the associated Legendre function P_m^m with the original eigenvalue E taking $m(m+1)$. Therefore, when $\alpha \neq 0$, the ground energy is also $-m^2$ and the ground eigenfunction is $\Theta(z, m, E)$, which must be nodeless. Actually the original eigenvalue E could be determined in the following by requiring the eigenfunction finite at the infinities. Its value could also be obtained from the previous papers [9], [10].

The super-potential W is determined from the potential $V(z, \alpha)$ by

$$W^2 - \frac{dW}{dz} = V(z, \alpha) + m^2 \tag{17}$$

where subtracting the ground eigenvalue $-m^2$ to make the potential factorable. This equation is the same difficult to treat as the Schrödinger original Eq.(15), hence, the perturbation method is used to solve it. First, the super-potential W is expanded as the series of the parameter α, that is,

$$W = W_0 + \alpha W_1 + \alpha^2 W_2 + \alpha^3 W_3 + \ldots = \sum_{n=0}^{\infty} \alpha^n W_n. \tag{18}$$

$$\begin{aligned} W^2 &- W' = W_0^2 - W_0' + \alpha\left(2W_0W_1 - W_1'\right) + \alpha^2\left(2W_0W_2 + W_1^2 - W_2'\right) \\ &+ \alpha^3\left(2W_0W_3 + 2W_1W_2 - W_3'\right) + \alpha^4\left(2W_0W_4 + 2W_1W_3 + W_2^2 - W_4'\right) + \ldots . \end{aligned} \tag{19}$$

Secondly, the original eigenvalue E must also be written as

$$\sum_{n=0}^{\infty} E_{0,n;m}\alpha^n \tag{20}$$

where there are three lower indices in the parameter $E_{0,n;m}$ with the index 0 refereing to the original ground state energy, and the other index n meaning its nth term expanded in the series in parameter α and the last m indicating the parameter $-m^2$ in Eq.(15). One can write the perturbation equation as

$$W^2 - W' = V(z,\alpha,m) + m^2 = -\left[E\mathrm{sech}^2 z + \alpha\mathrm{sech}^2 z - \alpha\mathrm{sech}^4 z\right] + m^2 \tag{21}$$

$$= -\left[\sum_{n=0}^{\infty} \alpha^n E_{0n}\mathrm{sech}^2 z + \alpha\mathrm{sech}^2 z - \alpha\mathrm{sech}^4 z\right] + m^2. \tag{22}$$

Comparing Equations (17), (19), and (22), one could get

$$W_0^2 - W_0' = -E_{0,0;m}\mathrm{sech}^2 z + m^2 \tag{23}$$

$$2W_0W_1 - W_1' = \mathrm{sech}^4 z - (E_{0,1;m}+1)\mathrm{sech}^2 z \tag{24}$$

$$2W_0W_2 + W_1^2 - W_2' = -E_{0,2;m}\mathrm{sech}^2 z \tag{25}$$

$$2W_0W_3 + 2W_1W_2 - W_3' = -E_{0,3;m}\mathrm{sech}^2 z \tag{26}$$

$$\ldots$$

$$2W_0W_n + \sum_{k=1}^{n-1} W_kW_{n-k} - W_n' = -E_{0,n;m}\mathrm{sech}^2 z \tag{27}$$

From Eq.(23), we get

$$W_0 = m\tanh z,\ E_{0,0;m} = m(m+1). \tag{28}$$

Then, we can write the other equations more concisely

$$W_1' - 2m\tanh zW_1 = (E_{0,1;m}+1)\mathrm{sech}^2 z - \mathrm{sech}^4 z \tag{29}$$

$$W_2' - 2m\tanh zW_2 = E_{0,2;m}\mathrm{sech}^2 z + W_1^2 \tag{30}$$

$$W_3' - 2m\tanh zW_3 = E_{0,3;m}\mathrm{sech}^2 z + 2W_1W_2 \tag{31}$$

$$\ldots$$

$$W_n' - 2m\tanh zW_n = E_{0,n;m} + \sum_{k=1}^{n-1} W_kW_{n-k} \tag{32}$$

$$\tag{33}$$

After obtaining the zero term W_0 for the super-potential W, the first order W_1 can be gotten as

$$W_1 = \bar{A}_1 \cosh^{2m} z \tag{34}$$

with

$$\begin{aligned}
\frac{d\bar{A}_1}{d\theta} &= \operatorname{sech}^{2m} z \left[(E_{0,1;m} + 1)\operatorname{sech}^2 z - \operatorname{sech}^4 z \right] && (35)\\
\bar{A}_1 &= \int \left[(E_{0,1;m} + 1)\operatorname{sech}^{2m+2} z - \operatorname{sech}^{2m+4} z \right] dz && (36)
\end{aligned}$$

In order to simplify the calculation, we just write some useful formula[19]

$$\begin{aligned}
Q(m, z) &= \int \operatorname{sech}^{2m} z dz \\
&= \frac{\sinh z}{2m - 1} \left[\operatorname{sech}^{2m-1} z + \sum_{k=1}^{m-1} \frac{2^k (m-1)(m-2)\ldots(m-k)\operatorname{sech}^{2m-2k-1}}{(2m-3)(2m-5)\ldots(2m-2k-1)} \right] && (37)
\end{aligned}$$

This formula could be written in concise form

$$Q(m, z) = \frac{\sinh z}{m} \sum_{k=0}^{m-1} I(m, k) \operatorname{sech}^{2m-2k-1} z, \tag{38}$$

where

$$I(m, k) = \frac{2^k m(m-1)(m-2)\ldots(m-k)}{(2m-1)(2m-3)(2m-5)\ldots(2m-2k-1)}. \tag{39}$$

Hence,

$$\begin{aligned}
Q(m + n, z) &= \int \operatorname{sech}^{2(m+n)z} dz \\
&= \frac{\sinh z}{m + n} \sum_{k=0}^{m+n-1} I(m + n, k) \operatorname{sech}^{2(m+n)-2k-1} z && (40)\\
&= Q_1(m + n, z) + 2(m + 1) I(m + n, n - 2) Q(m + 1, z) && (41)
\end{aligned}$$

where

$$Q_1(m + n, z) = \frac{\sinh z}{m + n} \sum_{j=1}^{n-1} I(m + n, n - 1 - j) \operatorname{sech}^{2m+2j+1} z \tag{42}$$

See the appendix 1 for details. Hence, the quantity $\bar{A}_1$ is

$$\begin{aligned}
\bar{A}_1 &= (E_{0,1;m} + 1) Q(m + 1, z) - Q(m + 2, z) \\
&= (E_{0,1;m} + 1 - 2(m + 1) I(m + 2, 0))\, Q(m + 1, z) - Q_1(m + 2, z) \\
&= a_1 Q(m + 1, z) - Q_1(m + 2, z) && (43)
\end{aligned}$$

where the introducing $a_1 = E_{0,1;m} + 1 - 2(m + 1) I(m + 2, 0)$ is used. So

$$\begin{aligned}
W_1 &= \bar{A}_1 \cosh^{2m} z \\
&= a_1 Q(m + 1, z) \cosh^{2m} z - Q_1(m + 2, z) \cosh^{2m} z && (44)
\end{aligned}$$

The quantity $E_{0,1;m}$ needs to be determined by the boundary conditions that the Θ is finite at $\pm\infty$. This in turn require to calculate the ground eigenfunction upon to the first order by

$$\begin{aligned}
\Theta_0 &= N \exp\left[-\int W dz\right] && (45)\\
&= N \exp\left[-\int W_0 dz \ -\alpha \int W_1 dz\right] * \exp O(\alpha^2) && (46)\\
&= N \mathrm{sech}^m z \exp\left[-\alpha \int W_1 dz\right] * \exp O(\alpha^2). && (47)
\end{aligned}$$

Whenever the eigenfunction is obtained, the boundary conditions finite $\Psi|_{\pm\infty}$ would choose the proper $E_{0,1;m}$. it is easy to compute

$$\int W_1 dz = \int a_1 Q(m+1,z)\cosh^{2m} z dz - \int Q_1(m+2,z)\cosh^{2m} z dz \qquad (48)$$

By the relations (38),(41),(42) it is easy to obtain the eigenfunction

$$\begin{aligned}
\Theta_0 &= N \mathrm{sech}^m z \exp\left[\alpha\left(a_1 \sum_{k=0}^{m} \frac{I(m+1,k)\cosh^{2k} z}{2k(m+1)}\right)\right]\\
&\quad * \exp\left[\alpha \frac{I(m+2,0)\mathrm{sech}^2 \mathrm{z}}{2m+4}\right] * \exp O(\alpha^2) && (49)
\end{aligned}$$

Unless the coefficient a_1 of the term $\sum_{k=0}^{m} \frac{I(m+1,k)\cosh^{2k} z}{2k(m+1)}$ becomes zero, the eigenfunction will blow up at infinity whenever $a_1\alpha < 0$,. So, $E_{0,1;m}$ is determined by $a_1 = E_{0,1;m} + 1 - 2(m+1)I(m+n,0) = 0$, the results are

$$\begin{aligned}
E_{0,1;m} &= -\frac{1}{2m+3} && (50)\\
W_1 &= -Q_1(m+2,z)\cosh^{2m} z = -\frac{\sinh z}{m+2} I(m+2,0)\mathrm{sech}^3 z\\
&= -\frac{\sinh z \mathrm{sech}^3 z}{2m+3} && (51)\\
\Theta_0 &= N \mathrm{sech}^m z \exp\left[\alpha \frac{\mathrm{sech}^2 \mathrm{z}}{4m+6}\right] * \exp O(\alpha^2). && (52)
\end{aligned}$$

With the first order term of the super-potential W_1, one could compute the second term W_2 by the same process.

$$W_2 = \bar{A}_2 \cosh^{2m} z \qquad (53)$$

with

$$\begin{aligned}
\bar{A}_2 &= \int \mathrm{sech}^{2m} z \left(E_{0,1;m}\mathrm{sech}^2 z + W_1^2\right) dz\\
&= E_{0,1;m} Q(m+1,z) + \int \mathrm{sech}^{2m} z \frac{\sinh^2 z \mathrm{sech}^6 z}{(2m+3)^2} dz\\
&= E_{0,1;m} Q(m+1,z) + \frac{1}{(2m+3)^2}\left(Q(m+2,z) - Q(m+3,z)\right)
\end{aligned}$$

$$\begin{aligned} &= \left[E_{0,1;m} + \frac{2(m+1)}{(2m+3)^2}\left(I(m+2,0) - I(m+3,1)\right)\right] Q(m+1,z) \\ &+\frac{1}{(2m+3)^2}\left[Q_1(m+2,z) - Q_1(m+3,z)\right] \end{aligned} \tag{54}$$

Similarly, the coefficient $a_2 = E_{0,1;m} + \frac{2(m+1)}{(2m+3)^2}\left(I(m+2,0) - I(m+3,1)\right)$ of the term $Q(m+1,z)$ must zero. The results are elegant:

$$E_{0,2;m} = -\frac{2m+2}{(2m+3)^3(2m+5)} \tag{55}$$

$$W_2 = \left[\frac{\sinh z \mathrm{sech}^3 z}{(2m+3)^3(2m+5)} - \frac{\sinh z \mathrm{sech}^5 z}{(2m+3)^2(2m+5)}\right]. \tag{56}$$

We can continue to calculate $W_3,\ W_4,\ \ldots,\ W_n,\ \ldots$. here we just use induction to prove that

$$W_n = \sinh z \sum_{k=1}^{n} a_{n,k} \mathrm{sech}^{2k+1} z \tag{57}$$

with $a_{n,k}$ could be determined by $a_{i,j}, i < n, j < n$, which appear in $W_i,\ i < n$. Obviously, W_1 in Eq.(51) satisfies the requirement (57) in the case $n = 1$. Suppose that $W_k,\ k < n$ meets the requirement, then

$$W_n = \bar{A}_n \cosh^{2m} \tag{58}$$

with $\bar{A_n}$ is determined by

$$\bar{A}_n = \int \mathrm{sech}^{2m} z \left[E_{0,n;m} \mathrm{sech}^2 z + \sum_{k=1}^{n-1} W_k W_{n-k}\right] dz. \tag{59}$$

The first thing is to simplify the following term:

$$\begin{aligned} \sum_{k=1}^{n-1} W_k W_{n-k} &= \sum_{k=1}^{n-1}\sum_{i=1}^{k}\sum_{j=1}^{n-k} a_{k,i} a_{n-k,j} \sinh^2 z \mathrm{sech}^{2(i+j)+2} z \\ &= \sum_{k=1}^{n-1}\sum_{p=2}^{n}\sum_{j=1}^{p-1} a_{k,p-j} a_{n-k,j} \sinh^2 z \mathrm{sech}^{2p+2} z \\ &= \sum_{k=1}^{n-1}\sum_{p=2}^{n}\sum_{j=1}^{p-1} a_{k,p-j} a_{n-k,j} \left[\mathrm{sech}^{2p} z - \mathrm{sech}^{2p+2} z\right] \\ &= \sum_{k=1}^{n-1}\sum_{p=3}^{n}\sum_{j=1}^{p-1} \left[a_{k,p-j} a_{n-k,j} - a_{k,p-1-j} a_{n-k,j}\right] \mathrm{sech}^{2p} z \\ &\quad + \sum_{k=1}^{n-1}\sum_{j=1}^{2} a_{k,2-j} a_{n-k,j} \mathrm{sech}^4 z - \sum_{k=1}^{n-1}\sum_{j=1}^{n} a_{k,n-j} a_{n-k,j} \mathrm{sech}^{2n+2} z \\ &= \sum_{p=2}^{n+1} b_{n,p} \mathrm{sech}^{2p} z \end{aligned} \tag{60}$$

$$b_{n,p} = \sum_{k=1}^{n-1}\sum_{j=1}^{p-1} \left[a_{k,p-j} a_{n-k,j} - a_{k,p-1-j} a_{n-k,j}\right], p = 2, 3, \ldots, n+1 \tag{61}$$

with the conditions $i < j,\ or j < 1$ then

$$a_{i,j} = 0 \tag{62}$$

$$\begin{aligned}
\bar{A}_n &= E_{0,n;m} Q(m+1,z) + \sum_{p=2}^{n+1} b_{n,p} Q(m+p,z) \\
&= \left[E_{0,n;m} + 2\sum_{p=2}^{n+1} b_{n,p} \frac{m+1}{m+p} I(m+p,p-2) \right] Q(m+1,z) \\
&\quad + \sum_{p=2}^{n+1} b_{n,p} Q_1(m+p,z)
\end{aligned} \tag{63}$$

In order to make the eigenfunction finite at infinity, the quantity $E_{0,m,n+1}$ is selected by

$$E_{0,n;m} + 2\sum_{p=2}^{n+1} b_{n,p}(m+1) I(m+p,p-2) = 0,$$

that is,

$$E_{0,n;m} = -2\sum_{p=2}^{n+1} b_{n,p}(m+1) I(m+p,p-2). \tag{64}$$

hence,

$$\begin{aligned}
W_n &= \bar{A}_n \cosh^{2m} z = \sum_{p=2}^{n+1} b_{n,p} Q_1(m+p,z) \cosh^{2m} z \\
&= \sum_{p=2}^{n+1} b_{n,p} \frac{\sinh z}{m+p} \sum_{j=1}^{p-1} I(m+p,p-1-j) \mathrm{sech}^{2j+1} z \\
&= \sinh z \sum_{j=1}^{n} \sum_{p=j+1}^{n+1} b_{n,p} \frac{I(m+p,p-1-j)}{m+p} \mathrm{sech}^{2j+1} z \\
&= \sinh z \sum_{j=1}^{n} a_{n,j} \mathrm{sech}^{2j+1} z
\end{aligned} \tag{65}$$

$$a_{n,j} = \sum_{p=j+1}^{n+1} b_{n,p} \frac{I(m+p,p-1-j)}{m+p} \tag{66}$$

we see that $a_{n,j}$ are determined by $b_{n,p}$, which are completely defined by $a_{i,j}, i < n, j < n$. this completes our proof. Therefore, the super-potential W could be written as

$$W = W_0 + \sum_{n=1}^{\infty} W_n \alpha^n = W_0 + \sum_{n=1}^{\infty} \sinh z \sum_{j=1}^{n} a_{n,j} \mathrm{sech}^{2j+1} z \alpha^n \tag{67}$$

The ground eigenfunction becomes

$$
\begin{aligned}
\Theta_0 &= N\exp\left[-\int W dz\right] \\
&= N\exp\left[-\int W_0 dz \ - \sum_{n=1}^{\infty}\alpha^n \int W_n dz\right] \\
&= N\exp\left[-\int m\tanh z dz - \int\sum_{n=1}^{\infty}\sum_{j=1}^{n}\alpha^n a_{n,j}\sinh z \mathrm{sech}^{2j+1} z dz\right] \\
&= N\mathrm{sech}^m z\exp\left[\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\alpha^n a_{n,j}}{2j}\mathrm{sech}^{2j} z\right]. \qquad (68)
\end{aligned}
$$

Back to the independent variable θ, $\mathrm{sech}z = \sin\theta$, the above ground eigenfunction becomes

$$
\Theta_0 = N\sin^m\theta\exp\left[-\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\alpha^n a_{n,j}}{2j}\sin^{2j}\theta\right] \qquad (69)
$$

The original ground eigen-energy is

$$
E_{0;m} = m(m+1) + \sum_{n=1}^{\infty}\alpha^n E_{0,n;m} \qquad (70)
$$

with $E_{0,n;m}$ is determined by Eq.(64).

Comparison with the former results

The elegant forms (69),(70),(64)for the ground eigenfunction and eigenvalue are the same as that in the reference [10]. There appeared two forms for the super-potential: $W(z)$ of the form (67) and $W(\theta)$ of (14) in the reference [10]. Whatever forms may the super-potentials be, they should give the same eigenfunctions for the spheroidal equations. Thus, it results the relation between them as

$$
\Theta_0(\theta) = \int W(z)\frac{dz}{d\theta}d\theta. \qquad (71)
$$

By $z = \lg\tan\frac{\theta}{2}$, it is easy to get

$$
W(\theta) = W(z)\frac{dz}{d\theta} = \frac{1}{\sin\theta}W(z). \qquad (72)
$$

Writing $W(z)$ back as the function of the original independent variable θ, one gets

$$
\begin{aligned}
W(\theta) &= W_0(\theta) + \sum_{n=1}^{\infty}W_n(\theta)\alpha^n \qquad (73) \\
W_0(\theta) &= -m\cot\theta, \qquad (74) \\
W_n(\theta) &= W_n = -\cos\theta\sum_{k=1}^{n}a_{n,k}\sin^{2k-1}\theta \qquad (75)
\end{aligned}
$$

Comparing the results with that (14) of the results of the reference [10], they are the same except for the $W_0(\theta)$. The difference between them is originated from the eigenfunction's relation

$$\Theta_0 = \frac{\Psi_0}{\sin^{\frac{1}{2}}\theta} \tag{76}$$

$$\Psi_0 = \int W(\theta)\theta. \tag{77}$$

where $W(\theta)$ are the forms of (14). The advantage of the method is that it can be easily extended to the spin-weighted spheroidal equations. It will be our further study.

Obviously non-shape-invariance property of the spheroidal functions

In order to check whether the spheroidal wave equations have the shape-invariance property, the super-potential W is rewritten as the following form:

$$W(A_{n,j}, z) = A_{0,0}m\tanh z + \sum_{n=1}^{\infty}\sinh z\sum_{j=1}^{n}A_{n,j}a_{n,j}\text{sech}^{2j+1}z\alpha^n, \tag{78}$$

With the definition

$$W_n(A_{i,j}, z) = \sinh z\sum_{j=1}^{n}A_{n,j}a_{n,j}\text{sech}^{2j+1}z \tag{79}$$

Then, $V^{\pm}(A_{n,j})$ are defined as

$$V^{\pm}(A_{n,j}, z) = W^2(A_{n,j}, z) \mp W' = \sum_{n=0}^{\infty}\alpha^n V_n^{\pm}(A_{i,j}, z). \tag{80}$$

We will check whether or not $V_n^{\pm}(A_{i,j}, z)$ are related with by the relations

$$V_n^{+}(A_{i,j}, z) = V_n^{-}(B_{i,j}, z) \tag{81}$$

step by step.

First, we write the special cases for $V_n^{\pm},\ n = 0,\ 1$. When $n = 0$,

$$V_0^- = W_0^2(A_{0,0}, z) - W_0^2(A_{0,0}, z) = m^2A_{0,0}^2\tanh^2 z - A_{0,0}m\text{sech}^2 z \tag{82}$$

$$= m^2A_{0,0}^2 - (m^2A_{0,0}^2 + A_{0,0}m)\text{sech}^2 z \tag{83}$$

$$V_0^+ = W_0^2(A_{0,0}, z) + W_0^2(A_{0,0}, z) = m^2A_{0,0}^2\tanh^2 z + A_{0,0}m\text{sech}^2 z \tag{84}$$

$$= m^2A_{0,0}^2 - (m^2A_{0,0}^2 - A_{0,0}m)\text{sech}^2 z \tag{85}$$

$$= V_0^-(B_{0,0}, z) = m^2B_{0,0}^2 - (m^2B_{0,0}^2 + B_{0,0}m)\text{sech}^2 z + R_0(A_{0,0}) \tag{86}$$

$$R_0(A_{0,0}) = m^2A_{0,0}^2 - m^2B_{0,0}^2 \tag{87}$$

Therefore

$$B_{0,0} = A_{0,0} - \frac{1}{m},\ R_0(A_{0,0}) = 2mA_{0,0} - 1. \tag{88}$$

This results is exact when $\alpha = 0$, and it just shows that the associated Legendre equations have the shape-invariance properties. It is easy to get the recurrence relations (4) for the associated Legendre functions from the the shape-invariance properties. What interests us most is whether this property could extend to the spheroidal functions, the case $\alpha \neq 0$.

When $n = 1$,

$$\begin{aligned}
V_1^-(A_{0,0}, A_{1,1}, z) &= 2W_0(A_{0,0})W_1(A_{1,1}, z) - W_1'(A_{1,1}, z) \\
&= -\frac{2mA_{0,0}A_{1,1}}{2m+3}\sinh^2 z\mathrm{sech}^4 z + \frac{A_{1,1}}{2m+3}\left[\sinh z\mathrm{sech}^3 z\right]' \\
&= \frac{(2mA_{0,0}+3)A_{1,1}}{2m+3}\mathrm{sech}^4 z + \frac{(2-2mA_{0,0})A_{1,1}}{2m+3}\mathrm{sech}^2 z \qquad (89) \\
V_1^+(A_{0,0}, A_{1,1}, z) &= 2W_0(A_{0,0})W_1(A_{1,1}, z) + W_1'(A_{1,1}, z) \\
&= -\frac{2mA_{0,0}A_{1,1}}{2m+3}\sinh^2 z\mathrm{sech}^4 z + \frac{A_{1,1}}{2m+3}\left[\sinh z\mathrm{sech}^3 z\right]' \\
&= \frac{(2mA_{0,0}-3)A_{1,1}}{2m+3}\mathrm{sech}^4 z + \frac{(-2-2mA_{0,0})A_{1,1}}{2m+3}\mathrm{sech}^2 z \qquad (90)
\end{aligned}$$

If we require the first term with the shape-invariance property, that is,

$$\begin{aligned}
V_1^+(A_{0,0}, A_{1,1}, z) &= V_1^-(B_{0,0}, B_{1,1}, z) \\
&= \frac{(2mB_{0,0}+3)B_{1,1}}{2m+3}\mathrm{sech}^4 z + \frac{(2-2mB_{0,0})B_{1,1}}{2m+3}\mathrm{sech}^2 z, \qquad (91)
\end{aligned}$$

there exist two conditions for the quantity $B_{1,1}$ to meet:

$$(2mB_{0,0}+3)B_{1,1} = (2mA_{0,0}-3)A_{1,1} \qquad (92)$$

$$(2-2mB_{0,0})B_{1,1} = (-2-2mA_{0,0})A_{1,1}. \qquad (93)$$

It is easy to see that these two conditions are not compatible, one is led to the conclusion the shape-invariance property is not hold for the spheroidal functions. first, we dot not admit them and investigate carefully. The shape-invariance property of the zero-term (88) shows the spheroidal functions have the same property as that of the associated Legendre functions. The new eigenvalue of the Eq.(15) is the quantity $-m^2$, whereas the original eigenvalue E now is contained in the expression $E\mathrm{sech}^2 z$. Therefore, when the new eigenvalue changes from $-m^2$ to $-(m-1)^2$, the original eigenvalue E can not remains unchanged. this is the great difference of the spheroidal functions from the the associated Legendre functions where the original eigenvalue remains the same as the new eigenvalue changes from $-m^2$ to $-(m-1)^2$. So the potential have no shape-invariance property.

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Appendix1: Simplification of the quantity $Q(m+n, z)$

$$\begin{aligned}
Q(m+n, z) &= \int \operatorname{sech}^{2(m+n)z} dz = \frac{\sinh z}{m+n} \sum_{k=0}^{m+n-1} I(m+n, k) \operatorname{sech}^{2(m+n)-2k-1} z \\
&= Q_1(m+n, z) + Q_2(m+n, z) \qquad (94)
\end{aligned}$$

where the two parts $Q_1(m+n, z), Q_2(m+n, z)$ are:

$$Q_1(m+n, z) = \frac{\sinh z}{m+n} \sum_{k=0}^{n-2} I(m+n, k) \operatorname{sech}^{2(m+n)-2k-1} z \qquad (95)$$

$$Q_2(m+n, z) = \frac{\sinh z}{m+n} \sum_{k=n-1}^{m+n-1} I(m+n, k) \operatorname{sech}^{2(m+n)-2k-1} z. \qquad (96)$$

In the above equation for the quantity Q_1, the change of $j = n-1-k$ simplifies it as

$$Q_1(m+n, z) = \frac{\sinh z}{m+n} \sum_{j=1}^{n-1} I(m+n, n-1-j) \operatorname{sech}^{2m+2j+1} z. \qquad (97)$$

The second part $Q_2(m+n, z)$is connected with $Q(m, z)$. Here is the proof. First changing k in the above equation for the quantity Q_2 to $p = k-(n-1)$, it is easy to obtain

$$\begin{aligned}
Q_2(m+n, z) &= \frac{\sinh z}{m+n} \sum_{p=0}^{m} I(m+n, p+(n-1)) \operatorname{sech}^{2m-2p+1} z \\
&= \frac{\sinh z}{m+n} \sum_{p=0}^{m} I(m+1+(n-1), p+(n-1)) \operatorname{sech}^{2m-2p+1} z \qquad (98)
\end{aligned}$$

here the relations between $I(m+l, k+l)$ and $I(m, k)$ are

$$\begin{aligned}
& I(m+l, k+l) \\
=\ & \frac{2^{k+l}(m+l)(m+l-1)\ldots m(m-1)(m-2)\ldots(m-k)}{(2m+2l-1)(2m+2l-3)\ldots(2m-1)(2m-3)(2m-5)\ldots(2m-2k-1)} \\
=\ & \frac{2*2^{l-1}(m+l)(m+l-1)\ldots(m+1)}{(2m+2l-1)(2m+2l-3)\ldots(2m+1)} \frac{2^k m(m-1)(m-2)\ldots(m-k)}{(2m-1)(2m-3)\ldots(2m-2k-1)} \\
=\ & 2I(m+l, l-1)I(m, k). \qquad (99)
\end{aligned}$$

So,

$$I(m+1+(n-1), p+(n-1)) = 2I(m+1+(n-1), n-2)I(m+1, p) \qquad (100)$$

The quantity $Q_2(m+n, z)$ becomes

$$\begin{aligned}
Q_2(m+n, z) &= \frac{\sinh z}{m+n} \sum_{p=0}^{m} 2I(m+1+(n-1), (n-2))I(m+1, p) \operatorname{sech}^{2m-2p+1} z \qquad (101) \\
&= 2I(m+1+(n-1), (n-2)) \frac{\sinh z}{m+n} \sum_{p=0}^{m} I(m+1, p) \operatorname{sech}^{2m-2p+1} z \\
&= 2I(m+n, n-2)(m+1)Q(m+1, z). \qquad (102)
\end{aligned}$$

Therefore,

$$Q(m+n,z)=Q_1(m+n,z)+2(m+1)I(m+n,n-2)Q(m+1,z). \tag{103}$$

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